Portfolio Theory & Risk

Portfolio risk as measured mathematically is the variance, or sqrt(Variance) referred to as standard deviation, of portfolio returns.

For a multi-stock portfolio with weightings w_i from w_1 to w_n and $\boldsymbol{\sigma_i}$ as stock i's standard deviation:

$$\sigma_p^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j>1}^{n} w_i \, w_j Covariance(\sigma_i, \sigma_j)$$

Let's simplify the above equation by assuming that we have an equal weighted stock portfolio. For an equal weighted stock portfolio we can substitute $1/n$ for w_i and w_j -- i.e. for a 10 stock equal weight portfolio each stock is weighted 1/10 or 0.1.

$$\sigma_p^2 = \sum_{i=1}^{n} \frac{1}{n^2} \sigma_i^2 + \sum_{i=1}^{n} \sum_{j>1}^{n} \frac{1}{n^2} Covariance(\sigma_i, \sigma_j)$$

Let's simplify further and assume that all stocks have the same Variance, $\sigma_i = \sigma_s$ and same Covariance. This is a big assumption, but as n gets large thanks to the law of large numbers the overall portfolio variance will reflect the average stock variance and average stock covariance. Since the summation $\sum_{i=1}^{n} \boldsymbol{\sigma_s}$ is equal to $n * \sigma_s$, substituting n for $\sum_{i=1}^{n}$ and $n-1$ for $\sum_{j>1}^{n}$ gets:

$$\sigma_p^2 = n \frac{1}{n^2} \sigma_s^2 + n(n-1) \frac{1}{n^2} Covariance$$

Cancelling out the n terms yields:

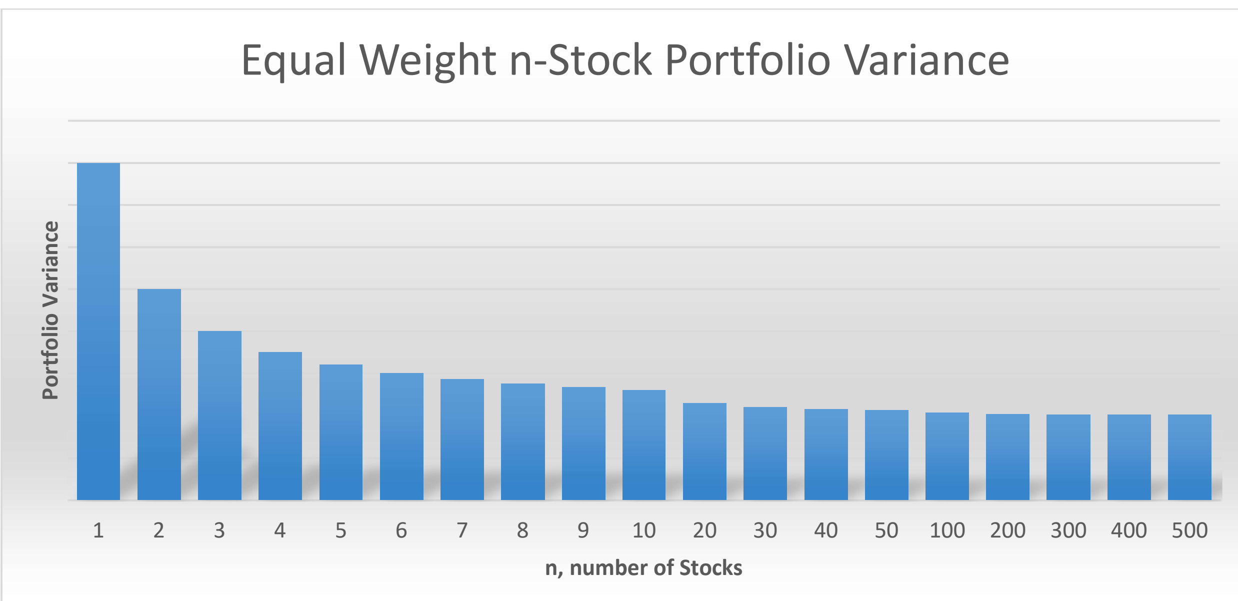
$$\sigma_p^2 = \frac{\sigma_s^2}{n} + \frac{(n-1)}{n} Covariance$$

From the above equation we can see that as n gets large the first term representing unsystematic risk, $\frac{\sigma_s^2}{n}$, is dominated by the second term representing systematic risk, $\frac{(n-1)}{n} Covariance$. Another way of saying this is that $\lim_{n\to\infty} \frac{\sigma_s^2}{n} = 0$ and $\lim_{n\to\infty} \frac{(n-1)}{n} Covariance = Covariance$.

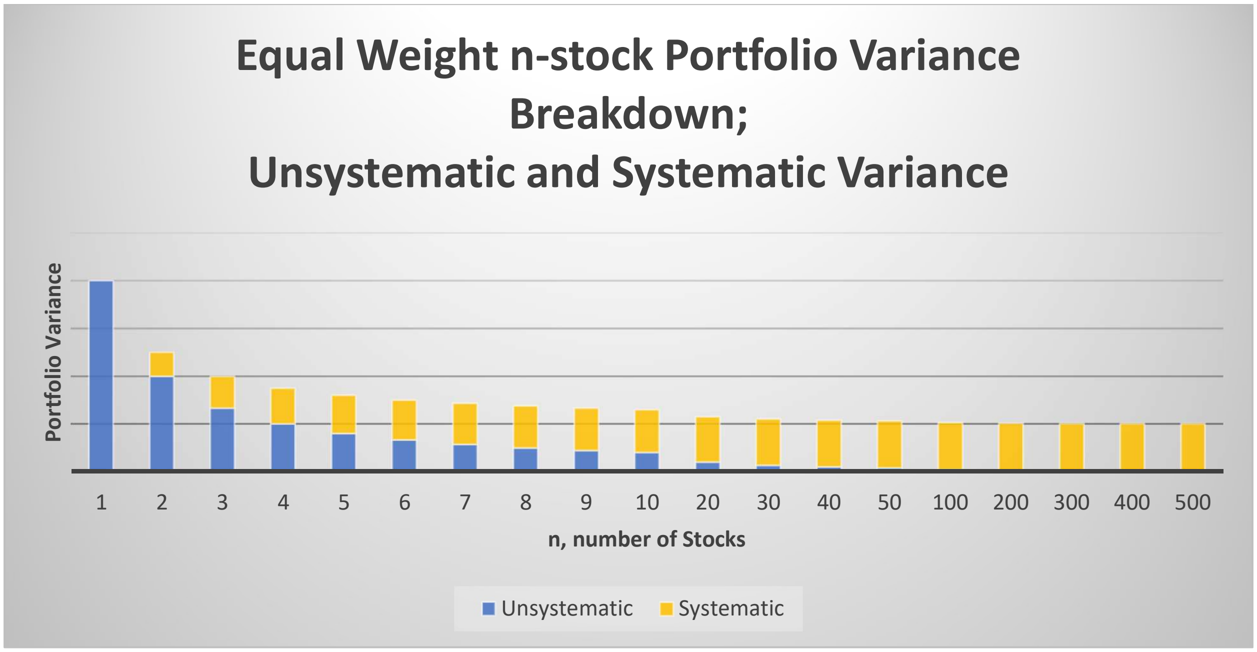
Therefore, for large *n*, $\sigma_p^2 \approx 0 + Covariance.$

From $\sigma_p^2 \approx 0 + Covariance$*, we can conclude that only the Covariance of holdings determines portfolio risk – **a key tenet of portfolio theory.*** Investors should be compensated for taking on risk (σ_p^2); the higher risk the higher the expected return. However, since single company unsystematic risk can be diversified away (0), investors should not be compensated with higher returns for taking on unsystematic risk. Only how much each individual stock co-varies with other stocks in the portfolio, determines how much portfolio risk there is for equal-weight portfolios with a large *n.*

When does *n* become large enough? With assumed stock variance of 400 and covariance of 100 for all stocks, you can see how increasing the number of stocks initially reduces portfolio variance (σ_p^2) dramatically, but each time *n* is increased the amount of the reduction in portfolio variance decreases as the portfolio variance limit of 100 is reached due to $\lim_{n\to\infty} \frac{(n-1)}{n} Covariance = Covariance.$



It is also helpful to see how portfolio variance is broken down between unsystematic risk – company-specific risk and systematic risk – undiversifiable market risk as *n* increases:



The Excel file that created the above charts is available for download http://alphamont.com/portfolio-theory/.